INVARIANTS WHICH ARE FUNCTIONS OF PARAMETERS OF THE TRANSFORMATION.*

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Among invariant theories of importance in geometry are those leading to concomitants which are functions of one or more of the coefficients of the transformations. An example is the case of invariants of axial rotations in a plane.† A general theory of such concomitants is here considered, for both binary and ternary linear transformations. What is usually true of a general analytical algorithm, ruling various special situations, is exemplified, i. e., the algorithm points to a natural mode of procedure in the investigation of any particular instance under the theory. Application is made in Section II to a theory of the invariants of relativity.

I. Theory of Binariants.

1. Concomitants which are irrational in the parameters. The transformation

$$T:\begin{cases} x_1 = \alpha_1 x_1' + \alpha_2 x_2', \\ x_2 = \beta_0 x_1' + \beta_1 x_2', \end{cases}$$

where the parameters are arbitrary, has two poles which are the roots of the respective linearly independent linear quantics‡

(1) $$f_{\pm 1} = 2\beta_0 x_1 + (\beta_1 - \alpha_1 \pm \Delta)x_2 \quad (\Delta = \sqrt{(\beta_1 - \alpha_1)^2 + 4\beta_0\alpha_2}).$$

These quantics are covariants of T, and they satisfy the invariant relations

(2) $$f_1' = \rho_1^{-1}f_1, \qquad f_{-1}' = \rho_{-1}^{-1}f_{-1},$$

in which

$$\rho_{\pm 1} = \tfrac{1}{2}(\beta_1 + \alpha_1 \pm \Delta), \qquad \rho_1\rho_{-1} = \alpha_1\beta_1 - \alpha_2\beta_0 \equiv D.$$

The binary form of order m

$$f = (a_0, a_1, \cdots, a_m)(x_1, x_2)^m$$

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† Cf. Boole, Cambridge Math. Journal, vol. 3 (1843); Elliott, *Algebra of Quantics* (First ed.), Chap. 15.

‡ Transactions Amer. Math. Society, vol. 18 (1917), p. 450.

has a unique expansion in terms of f_1, f_{-1} as arguments:

$$f = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i} f_1^{m-i} f_{-1}^{i}, \tag{3}$$

concerning which we prove the following

THEOREM. *The linearly independent functions* $\varphi_{m-2i}(i = 0, \cdots, m)$, *linear in* $a_0, \cdots, a_m$, *taken with* f_1, f_{-1}, *compose a complete system of concomitants of* f *under* T *in the domain* $R(1, T, \Delta)$ *to which* f_1, f_{-1} *belong.*

The independence of the expressions φ is evident. For, were they linearly connected, so would be the $m + 1$ linear functions of them, $a_0, \cdots, a_m$, but these are arbitrary.

Let f', the transformed of f by T, be expanded in terms of the arguments

$$f_{\pm 1}' = 2\beta_0 x_1' + (\beta_1 - \alpha_1 \pm \Delta) x_2';$$

$$f' = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i}' f_1'^{m-i} f_{-1}'^{i}.$$

Then the function φ_{m-2i}' is evidently the same function of $a_0', \cdots, a_m'$ that φ_{m-2i} is of $a_0, \cdots, a_m$; moreover, if we apply the inverse of T to f' and use the relations (2) we get

$$f = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i}' \rho_1^{2i-m} D^{-i} f_1^{m-i} f_{-1}^{i},$$

an expansion which must be identical with (3) since (3) is unique. Hence the expressions φ_{m-2i} are invariants of f under T satisfying the invariant relations

$$\varphi_{m-2i}' = \rho_1^{m-2i} D^i \varphi_{m-2i} \quad (i = 0, \cdots, m).$$

Any concomitant of f under T can be expressed in terms of $f_1, f_{-1}, \varphi_{m-2i}$ $(i = 0, \cdots, m)$ by means of (1) and the inverse of the following $m + 1$ linear substitutions on $a_0, \cdots, a_m$:

$$\varphi_{m-2i} = \varphi_{m-2i}(a_0, a_1, \cdots, a_m) \quad (i = 0, \cdots, m),$$

hence the theorem is proved.

2. Systems belonging to the domain of rational polynomials. If we seek a fundamental system of concomitants which belong to the domain $R(1, T, 0)$ of rational polynomials in $a_0, \cdots, a_m, x_1, x_2$ and the parameters of the transformation, we will be concerned with linear expressions in terms such as

$$I_r = \varphi_{m-2i_1}^{r_{i_1}} \varphi_{m-2i_2}^{r_{i_2}} \cdots f_1^{s_1} f_{-1}^{s_2}.$$

The invariant relation for I_r is

$$I_r' = \rho_1^{\Sigma r_i(m-2i) - s_1 + s_2} D^{\Sigma r_i i - s_2} I_r, \tag{4}$$

and a linear expression ψ in terms I_r is a concomitant if and only if $\Sigma r_i(m - 2i) - s_1 + s_2$ as well as $\Sigma r_i i - s_2$ are the same for every term. A necessary condition that ψ belong to $R(1, T, 0)$ is that, for each of its terms,

$$
(5)\qquad \begin{aligned} \nu &= \Sigma r_i(m - 2i) - s_1 + s_2 \\ &= r_{i_1}(m - 2i_1) + r_{i_2}(m - 2i_2) + \cdots - s_1 + s_2 = 0. \end{aligned}
$$

This amounts to a sufficient condition also, for, although not all products I_r are rational, those for which $\nu = 0$ can be arranged in conjugate pairs (I_r, I_{-r}) and the power of D in the invariant relation for I_{-r} will be the same as the power for I_r. In fact,

$$
I_{-r} = \varphi_{-(m-2i_1)}{}^{r_{i_1}}\varphi_{-(m-2i_2)}{}^{r_{i_2}} \cdots f_1{}^{s_2}f_{-1}{}^{s_1},
$$

for which the power of D is $D^{\Sigma r_i(m-i)-s_1}$. But, if $\nu = 0$,

$$
\Sigma r_i(m - i) - s_1 = \Sigma r_i i - s_2.
$$

We can now replace the terms of the pair (I_r, I_{-r}) by $I_r + I_{-r}$, $I_r - I_{-r}$, respectively, and the latter binomials belong, essentially, to $R(1, T, 0)$.

Any concomitant multinomial is here reducible in terms of invariant monomials. The question of the finiteness of a complete system, as well as the problem of determining it explicitly are, therefore, solved by a known lemma due to Hilbert, viz.: If an infinite system of monomials in n letters be formed according to any law sufficiently definite to locate an arbitrarily chosen monomial within or without the system, then there will always exist, within the system, a finite set of monomials such that every monomial of the system is divisible by at least one of the set.

In the present case the letters involved are φ_{m-2i} $(i = 0, \cdots, m)$, f_1, f_{-1}, and $n = m + 3$, while the law by which the system is formed is embodied in the linear diophantine equation

$$
(6)\qquad \begin{aligned} &s_2 + r_0 m + r_1(m - 2) + r_2(m - 4) + \cdots \\ &\qquad = \cdots + r_{m-2}(m - 4) + r_{m-1}(m - 2) + r_m m + s_1, \end{aligned}
$$

which is to be satisfied in positive integers (including zero) r_i, s_j. The terms on the two sides of this equation, adjacent to the equality sign, are $2r_{\frac{1}{2}m-1}$, $2r_{\frac{1}{2}m+1}$ if m is even, and $r_{\frac{1}{2}(m-1)}$, $r_{\frac{1}{2}(m+1)}$ if m is odd, and in the former case φ_0 exists and is included in the irreducible system.

We have now proved the following

THEOREM. *A fundamental system of concomitants of f under T in $R(1, T, 0)$ is given by the irreducible solutions of the linear diophantine equation* (6). *The number of concomitants in the system is equal to the finite number of these irreducible solutions, increased, if m is even, by unity.*

The enumeration of particular complete systems for special values of m is the same for the situation treated in section II as in the present general theory, and these details are omitted to be taken up in that connection. We add, also, that this theory holds if the quantity under the radical Δ is a square, so that Δ is only apparently an irrationality, but does not hold if $\Delta = 0$.

II. The Instance of Einstein's Relativity Transformations.

1. The transformations. Two moving systems of reference S and S' are conceived, which, for the sake of concreteness, may be taken to be two platforms on each of which are installed instruments for tàking measurements, such as clocks for measuring time, rules for measuring lengths, and so on. Let these two systems have the relative velocity v in the line l. Suppose that systems of rectangular coördinates are attached to S and S' in such a way that the x-axis of each system is in the line l, and let the y-axis and the z-axis on S be parallel, respectively, to the y'-axis and the z'-axis on S'. Supposing the origins to coincide at the time $t = 0$, let the coördinates on S be denoted by x, y, z, t, and those referring to S' by x', y', z', t'. Then, as was shown first by Einstein,

$$(7) \qquad t = \mu(c^2t' + vx')/c, \quad x = \mu(vt' + x')c, \quad y = y', \quad z = z',$$

where

$$\mu = 1/\sqrt{c^2 - v^2},$$

and c is the velocity of light.*

Let the first two equations of (7) be denoted by τ, then τ is unitary, and when we treat its invariant theory as a special situation under section I we have $D = 1$. Instead of distinguishing the two types of concomitants with respect to domains of rationality we now refer to them merely as non-absolute (relative) systems and absolute systems.

When (1) and (2) are particularized to correspond to τ we find that f_1, f_{-1} become, essentially,

$$(8) \qquad \xi = ct + x, \qquad \eta = ct - x.$$

These are universal covariants of τ for all values of the relative velocity v, for which the invariant relations are

$$\xi' = \rho^{-1}\xi, \qquad \eta' = \rho\eta,$$

where

$$\rho = \sqrt{\frac{c+v}{c-v}}.$$

* Einstein, Annalen der Physik, vol. 17 (1905). Lorentz, Einstein and Minkowski, *Das Relativitätsprinzip* (1913), p. 27. R. D. Carmichael, *The Theory of Relativity* (1913), p. 44.

Let

$$f = a_0 t^m + m a_1 t^{m-1} x + \cdots + a_m x^m,$$

where $a_0, \cdots, a_m$ are constants, or arbitrary functions of c, y, z, then the expansion (3) is given by substituting, in f, from the inverse of (8), i. e.,

$$(9) \qquad t = (\xi + \eta)/2c, \qquad x = c(\xi - \eta)/2c.$$

As it is evident that φ_{m-2i} contains the constant $(2c)^m$ in the denominator we write

$$(2c)^m \varphi_{m-2i} = \psi_{m-2i},$$

whence follows

$$\psi_{m-2i}' = \rho^{m-2i} \psi_{m-2i} \; (i = 0, \cdots, m).$$

The concomitants are here functions of the parameters in τ but they are, in fact, free from the relative velocity of the systems of reference.

We note that these systems become orthogonal invariant systems when the velocity of light c is replaced by $\sqrt{-1}$, the covariant $-\xi\eta$ then becoming the *absolute*, $x^2 + t^2$.

2. Calculation of the non-absolute system of f. The invariants ψ_{m-2i} can be derived in explicit form when m is general. Noting that

$$(\xi + \eta)^h (\xi - \eta)^k = \sum_{i=0}^{h+k} \left[\sum_{t=0}^{i} (-1)^t \binom{h}{i-t} \binom{k}{t} \right] \xi^{h+k-i} \eta^i,$$

it is evident that the transformed of

$$f = \sum_{j=0}^{m} \binom{m}{j} a_j t^{m-j} x^j,$$

by (9), is

$$(10) \qquad f = \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{t=0}^{i} (-1)^t \binom{m}{j} \binom{m-j}{i-t} \binom{j}{t} a_j c^j \xi^{m-i} \eta^i / (2c)^m.$$

Hence

$$(11) \quad \binom{m}{i} \psi_{m-2i} = \sum_{j=0}^{m} \sum_{t=0}^{i} (-1)^t \binom{m}{j} \binom{m-j}{i-t} \binom{j}{t} a_j c^j \quad (i = 0, \cdots, m).$$

A few special cases of this formula are added to facilitate writing down particular systems. The invariant $\psi_{-(m-2i)}$ may be obtained from ψ_{m-2i} by changing the signs of all odd powers of c in the latter, i. e., the ψ's at equal distances from the ends of expansion (10) are conjugates. Hence the four ψ's written below suffice to give at once the explicit complete systems for all orders up to $m = 7$ inclusive.

$$\psi_m = a_0 + m a_1 c + \binom{m}{2} a_2 c^2 + \cdots + \binom{m}{j} a_j c^j + \cdots + a_m c^m,$$

$$1! \binom{m}{1} \psi_{m-2} = m a_0 + \binom{m}{1} (m-2) a_1 c + \binom{m}{2} (m-4) a_2 c^2 + \cdots$$
$$+ \binom{m}{j} (m - 2j) a_j c^j + \cdots - m a_m c^m,$$

$$2!\binom{m}{2}\psi_{m-4} = m(m-1)a_0 + \binom{m}{1}(m-1)(m-4)a_1c$$
$$+ \binom{m}{2}(m^2 - 9m + 16)a_2c^2 + \cdots$$
$$+ \binom{m}{j}(m^2 - \overline{4j+1}\,m + 4j^2)a_jc^j + \cdots$$
$$+ m(m-1)a_mc^m,$$

$$3!\binom{m}{3}\psi_{m-6} = m(m-1)(m-2)a_0 + \binom{m}{1}(m-1)(m-2)(m-6)a_1c + \cdots$$
$$+ \binom{m}{j}[m^3 - 3(2j+1)m^2 + 2(6j^2 + 3j + 1)m$$
$$- 4(2j^3 + j)]a_jc^j + \cdots - m(m-1)(m-2)a_mc^m,$$

et cetera.

3. Absolute invariants of relativity. A complete system of absolute concomitants of f under τ is constructed from the irreducible solutions of (6). Thus, if $m = 2$, the sets of values of s_2, r_0, r_2, s_1 respectively, in these solutions, are 1, 0, 0, 1; 0, 1, 1, 0; 0, 1, 0, 2; 2, 0, 1, 0. Hence the system for the quadratic is

$$\gamma : \psi_0, \quad J = \xi\eta, \quad \delta : \psi_2\psi_{-2}, \quad \epsilon_{\pm 1} : \psi_2\xi^2 \pm \psi_{-2}\eta^2, \tag{12}$$

where a colon is used instead of the equality sign to indicate that irrelevant constant factors are to be deleted.

We arrange these solutions in tables as below (cf. table for $m = 2$), juxtaposing under a double notation, like $\epsilon_{\pm 1}$, the solutions representing conjugate products. If a solution is symmetrically placed with reference to the median line of the table the corresponding product is its own conjugate. We omit writing the systems in the form (12) as all types can easily be read off from the tables.

The number of concomitants in the absolute system of a quartic is 12.

$m = 1.$

	s_2	r_0	r_1	s_1
J	1	0	0	1
α	0	1	1	0
$\beta_{\pm 1}$	0	1	0	1
	1	0	1	0

$m = 2.$

	s_2	r_0	r_2	s_1
J	1	0	0	1
δ	0	1	1	0
$\epsilon_{\pm 1}$	0	1	0	2
	2	0	1	0

$m = 3.$

	s_2	r_0	r_1	r_2	r_3	s_1
J	1	0	0	0	0	1
ζ	0	1	0	0	1	0
η	0	0	1	1	0	0
$\theta_{\pm 1}$	0	1	0	3	0	0
	0	0	3	0	1	0
$\iota_{\pm 1}$	0	0	1	0	0	1
	1	0	0	1	0	0
$\kappa_{\pm 1}$	0	1	0	0	0	3
	3	0	0	0	1	0
$\lambda_{\pm 1}$	0	1	0	1	0	2
	2	0	1	0	1	0
$\mu_{\pm 1}$	0	1	0	2	0	1
	1	0	2	0	1	0

The actual invariants and covariants represented in these tables are ow given (cf. 11 and 12).

$m = 1.$

$$J = c^2t^2 - x^2, \quad \alpha = a_0^2 - c^2a_1^2, \quad \beta_{+1} = f, \quad \beta_{-1} = c^2a_1t + a_0x.$$

$m = 2.$

$$J, \quad \gamma = a_0 - c^2a_2, \quad \delta = a_0^2 - 4c^2a_1^2 + c^4a_2^2 + 2c^2a_0a_2,$$

$$\epsilon_{+1} = (a_0 + c^2a_2)(c^2t^2 + x^2) + 4c^2a_1tx,$$

$$\epsilon_{-1} = a_1(c^2t^2 + x^2) + (a_0 + c^2a_2)tx.$$

$$m = 3.$$

$$\begin{aligned}
J,\ \zeta &= a_0^2 - 9c^2a_1^2 + 9c^4a_2^2 - c^6a_3^2 + 6c^2a_0a_2 - 6c^4a_1a_3,\\
\eta &= a_0^2 - c^2a_1^2 + c^4a_2^2 - c^6a_3^2 - 2c^2a_0a_2 + 2c^4a_1a_3,\\
\theta_{\pm 1} &= (a_0 + 3ca_1 + 3c^2a_2 + c^3a_3)(a_0 - ca_1 - c^2a_2 + c^3a_3)^3\\
&\qquad \pm (a_0 - 3ca_1 + 3c^2a_2 - c^3a_3)(a_0 + ca_1 - c^2a_2 - c^3a_3)^3,\\
\iota_{+1} &= (a_0 - c^2a_2)t + (a_1 - c^2a_3)x,\\
\iota_{-1} &= (a_1 - c^2a_3)c^2t + (a_0 - c^2a_2)x,\\
\kappa_{+1} &= (a_0 + 3c^2a_2)(c^2t^3 + 3tx^2) + (3a_1 + c^2a_3)(3c^2t^2x + x^3),\\
\kappa_{-1} &= (3a_1 + c^2a_3)(c^4t^3 + 3c^2tx^2) + (a_0 + 3c^2a_2)(3c^2t^2x + x^3),\\
\lambda_{+1} &= A(c^2t^2 + x^2) + 4Bc^2tx, \quad \lambda_{-1} = B(c^2t^2 + x^2) + Atx,\\
\mu_{+1} &= Ct + Dx, \quad \mu_{-1} = Dc^2t + Cx,
\end{aligned}$$

where

$$\begin{aligned}
A &= a_0^2 - 3c^2a_1^2 - 3c^4a_2^2 + c^6a_3^2 + 2c^2a_0a_2 + 2c^4a_1a_3,\\
B &= a_0a_1 + c^2a_0a_3 - 3c^2a_1a_2 + c^4a_2a_3,\\
C &= a_0^3 + 3c^6a_2^3 - 5c^2a_0a_1^2 - 5c^4a_0a_2^2 + 3c^6a_0a_3^2 + c^2a_0^2a_2 + 9c^4a_1^2a_2\\
&\qquad + c^8a_2a_3^2 + 2c^4a_0a_1a_3 - 10c^6a_1a_2a_3,\\
D &= a_0^2a_1 + 3c^2a_0^2a_3 + 9c^4a_1a_2^2 - 5c^4a_1^2a_3 + c^6a_1a_3^2 - 5c^6a_2^2a_3\\
&\qquad + 2c^4a_0a_2a_3 - 10c^2a_0a_1a_2 + 3c^2a_1^3 + c^8a_3^3.
\end{aligned}$$

A single syzygy connects the quantics of each of the first two of these systems, as follows:

$$\Sigma_1 = \alpha J - c^2f^2 + \beta_{-1}^2 = 0,$$
$$\Sigma_2 = \delta J^2 - \epsilon_{+1}^2 + 4c^2\epsilon_{-1}^2 = 0.$$

III. Ternariants.

The general ternary transformations

$$S : \begin{cases} x = l_1x' + l_2y' + l_3z',\\ y = m_1x' + m_2y' + m_3z',\\ z = n_1x' + n_2y' + n_3z', \end{cases}$$

have three poles in a plane, and the linear ternary quantics representing the lines joining these poles in pairs, are covariants of S. In fact, assuming

that S transforms

$$f = a_1x + a_2y + a_3z \tag{13}$$

into kf' we readily find that k is a root of the characteristic equation

$$k^3 - \Sigma_1k^2 + \Sigma_2k - D = 0, \tag{14}$$

where Σ_i $(i = 1, 2)$ is the sum of all of the principal minors of order i, of the determinant D of S.

1. Invariants under rotations of three-dimensional axes. If S is the transformation which rotates a set of rectangular axes in the three-space into another rectangular system with the same origin, then, l_j, m_j, n_j are direction cosines connected by a variety of well-known relations. Then $D = \pm 1$, and the equation (14) can be reduced to

$$k^3 - ak^2 + \sigma ak - \sigma = 0, \tag{15}$$

where $\sigma = \operatorname{sgn} D$; i. e., is $+ 1$ or $- 1$ according as D is $+ 1$ or $- 1$; and a is the sum of the direction cosines occurring in the principal diagonal of D. Without loss of generality we can now substitute

$$a = \sigma + 2\sigma \cos \theta,$$

θ being an auxiliary angle, whence the three roots of (15) become σ, $\sigma e^{i\theta}$, $\sigma e^{-i\theta}$ $(i = \sqrt{-1})$. Replacing k, in $kf' = f$, by these three values, in succession, and solving for the ratios $a_1 : a_2 : a_3$, we find the three linear covariants of S to be

$$\begin{aligned} f_{\pm 1}^{(\sigma)} &= (l_3 + \sigma n_1 e^{\pm i\theta})x + (m_3 + \sigma n_2 e^{\pm i\theta})y + (n_3 - \overline{l_1 + m_2}\sigma e^{\pm i\theta} + e^{\pm 2i\theta})z, \\ f_0^{(\sigma)} &= (l_3 + \sigma n_1)x + (m_3 + \sigma n_2)y + (n_3 - \overline{l_1 + m_2}\sigma + 1)z. \end{aligned} \tag{16}$$

Three linear contravariants, representing the poles, are the eliminants of these three covariants taken, in pairs, with $ux + vy + wz$, but contragrediency is equivalent to cogrediency under S.

The general ternary quantic of order m,

$$f(x, y, z) = \Sigma \frac{\lfloor m}{\lfloor r \lfloor s \lfloor t} a_{rst}x^ry^sz^t \quad (r + s + t = m),$$

has a unique expansion* in terms of $f_{+1}^{(\sigma)}$, $f_{-1}^{(\sigma)}$, $f_0^{(\sigma)}$ as argument forms:

$$f = \sum_{t=0}^{m} \binom{m}{t} \sum_{i=0}^{m-t} \binom{m-t}{i} \varphi_{m-t-2i}^{(t,\sigma)} f_{+1}^{(\sigma)\,m-t-i} f_{-1}^{(\sigma)\,i} f_0^{(\sigma)\,t}.$$

The $\frac{1}{2}(m + 1)(m + 2)$ coefficient forms φ are invariants of f under S, linear in the coefficients a_{rst}, and belonging to the domain of complex

* Transactions Amer. Math. Society, vol. 15 (1914), p. 82.

numbers. They satisfy the invariant relations

$$\varphi_{m-t-2i}^{(t,\sigma)'} = \rho^{m-t-2i}\varphi_{m-t-2i}^{(t,\sigma)} \quad \begin{pmatrix} t = 0, \cdots, & m \\ i = 0, \cdots, & m-t \end{pmatrix}, \tag{17}$$

in which $\rho = \sigma e^{i\theta}$. The theorem below follows:

THEOREM. *A complete system of relative concomitants of f, under the transformations S of determinant σ, is composed of $f_{\pm 1}^{(\sigma)}$, $f_0^{(\sigma)}$, and the $\frac{1}{2}(m+1)(m+2)$ invariants $\varphi_{m-t-2i}^{(t,\sigma)}$.*

In order to construct explicitly the invariants (17), it is convenient to solve for the inverse of the system of equations (16) and substitute the resulting linear expressions in $f_{\pm 1}^{(\sigma)}$, $f_0^{(\sigma)}$, which are the values of x, y, z, in f.

The determinant of $f_{+1}^{(\sigma)}$, $f_{-1}^{(\sigma)}$, $f_0^{(\sigma)}$ can be put in the form

$$\Delta = \sigma(l_3n_2 - m_3n_1)(1-e^{i\theta})(1-e^{-i\theta})(e^{-i\theta}-e^{i\theta}),$$

and the solution of (16), and use of relations among direction cosines, gives

$$\begin{aligned} \Delta x &= A_{-1}f_{+1}^{(\sigma)} - A_{+1}f_{-1}^{(\sigma)} + A_0f_0^{(\sigma)}, \\ \Delta y &= B_{-1}f_{+1}^{(\sigma)} - B_{+1}f_{-1}^{(\sigma)} + B_0f_0^{(\sigma)}, \\ \Delta z &= C_{-1}f_{+1}^{(\sigma)} - C_{+1}f_{-1}^{(\sigma)} + C_0f_0^{(\sigma)}, \end{aligned}$$

in which

$$\begin{aligned} A_{\pm 1} &= (e^{\pm i\theta}-1)[\sigma(l_1m_3 - l_2l_3) - \sigma n_2e^{\pm i\theta} - m_3(e^{\pm i\theta}+1)], \\ B_{\pm 1} &= -(e^{\pm i\theta}-1)[\sigma(m_2l_3 - m_1m_3) - \sigma n_1e^{\pm i\theta} - l_3(e^{\pm i\theta}+1)], \\ C_{\pm 1} &= (e^{\pm i\theta}-1)\sigma(m_3n_1 - l_3n_2), \\ A_0 &= (e^{i\theta}-e^{-i\theta})[\sigma(l_1m_3 - l_2l_3) - \sigma n_2 - m_3(e^{i\theta}+e^{-i\theta})], \\ B_0 &= -(e^{i\theta}-e^{-i\theta})[\sigma(m_2l_3 - m_1m_3) - \sigma n_1 - l_3(e^{i\theta}+e^{-i\theta})], \\ C_0 &= (e^{i\theta}-e^{-i\theta})\sigma(m_3n_1 - l_3n_2). \end{aligned}$$

Through substitution in f, we obtain, by a well-known principle,

$$\varphi_m^{(0,\sigma)} = f(A_{-1}, B_{-1}, C_{-1})/\Delta^m,$$

and hence

$$\varphi_{m-t-2i}^{(t,\sigma)} = \Delta_1^i\Delta_0^t\varphi_m^{(0,\sigma)} \Big/ \frac{\lfloor m}{\lfloor m-i-t} \quad \begin{pmatrix} t = 0, \cdots, m \\ i = 0, \cdots, m-t \end{pmatrix},$$

where

$$\Delta_1 = -A_{+1}\frac{\partial}{\partial A_{-1}} - B_{+1}\frac{\partial}{\partial B_{-1}} - C_{+1}\frac{\partial}{\partial C_{-1}},$$

$$\Delta_0 = A_0\frac{\partial}{\partial A_{-1}} + B_0\frac{\partial}{\partial B_{-1}} + C_0\frac{\partial}{\partial C_{-1}}.$$

2. **Absolute concomitants.** For a fixed value of t $\varphi_{-(m-t-2i)}^{(t,\sigma)}$ is conjugate to $\varphi_{m-t-2i}^{(t,\sigma)}$ $(i = 0, \cdots, m-t)$; hence if

$$I_r = \prod_{t=0}^{m} \varphi_{m-t-2i_1}^{(t,\sigma)^{r_{i_1}^{(t)}}} \varphi_{m-t-2i_2}^{(t,\sigma)^{r_{i_2}^{(t)}}} \cdots f_{+1}{}^{(\sigma)^{s_1}} f_{-1}{}^{(\sigma)^{s_2}} f_0{}^{(\sigma)^{s_3}}$$

is any product from the relative system, the conjugate product is

$$I_{-r} = \prod_{t=0}^{m} \varphi_{-(m-t-2i_1)}^{(t,\sigma)^{r_{i_1}^{(t)}}} \varphi_{-(m-t-2i_2)}^{(t,\sigma)^{r_{i_2}^{(t)}}} \cdots f_{+1}{}^{(\sigma)^{s_2}} f_{-1}{}^{(\sigma)^{s_1}} f_0{}^{(\sigma)^{s_3}},$$

and the exponent of ρ in the invariant relation for I_r is

$$E = \sum_{t=0}^{m} \sum_{i=0}^{m-t} r_i^{(t)}(m - t - 2i) - s_1 + s_2.$$

The concomitants $I_r \pm I_{-r}$, deprived of constant factors, are both real. Application of Hilbert's lemma to the theory of absolute concomitants, as in I, 2, gives, therefore:

THEOREM. *A fundamental system of absolute concomitants of f, under the rotational transformations of determinant σ, is given by the finite set of irreducible solutions in positive integers $r_i^{(t)}$, s_j (zero values being included) of the linear diophantine equation $E = 0$. The number of concomitants in the system is equal to the number of irreducible solutions increased by $\frac{1}{2}(m + 4)$ if m is even and by $\frac{1}{2}(m + 3)$ if m is odd.*

The numbers $\frac{1}{2}(m + 4)$, $\frac{1}{2}(m + 3)$, added to the number of irreducible solutions, correspond to the existent absolute invariants $f_0{}^{(\sigma)}$, $\varphi_0{}^{(t,\sigma)}$, where t is such that $m - t$ is even. These are not furnished by the solution of $E = 0$.

The systems considered in the present section III may be interpreted as invariant systems of the curve $f = 0$ where f is transformed by the ternary substitutions furnished by the formulas for axial rotations in three-space. They may also be interpreted, in the sense of Boole, as systems of invariants of the three-dimensional surface

$$f(x, y, z) = 1,$$

under the operations of rotation of rectangular coördinate axes.

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